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COMPLETE INTERSECTION CALABI-YAU MANIFOLDS II
THREE GENERATION MANIFOLDS

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ABSTRACT

A class of Calabi-Yau spaces of Euler number $-6k$, for many values of k, is searched to find manifolds which admit a freely acting discrete symmetry group $\mathcal{G}$ of order k. For such manifolds $\mathcal{M}$ one may construct the quotient $\mathcal{M}/\mathcal{G}$ which is a Calabi-Yau manifold of Euler number -6 corresponding to three generations of particles. Surprisingly we are able to eliminate all but three of the manifolds. Of these one is the manifold constructed by Tian and Yau. The other two possibilities are manifolds which have $\chi = -48$. However we have been unable to find an explicit realization of groups of order eight. We correct also some assertions made in a recent article by Aspinwall *et al.*

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I. INTRODUCTION

In a recent publication [1] we have described the construction of an exhaustive catalogue of complete intersection Calabi-Yau manifolds (CICY manifolds). That is Calabi-Yau manifolds that may be realized as a complete intersection of polynomials in a Cartesian product of complex projective spaces. These manifolds generalize the Tian-Yau manifold [2] the covering manifold of which is realized in $P_3 \times P_3$ by three polynomial equations

$$\begin{aligned}
\sum_{A=0}^{3} x^A y^A &= 0 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\sum_{A=0}^{3} (x^A)^3 &= 0 \qquad \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\
\sum_{A=0}^{3} (y^A)^3 &= 0 \qquad \begin{bmatrix} 0 \\ 3 \end{bmatrix}
\end{aligned} \tag{1.1}$$

where x^A and y^B denote the homogeneous coordinates of the two P_3's. As explained in [1] the manifold is fully specified by the degree vectors that give the degrees of each of the polynomials in the variables of each projective space. These are written in the right hand column of equation (1.1). To establish notation we denote this manifold by a matrix whose columns are the degree vectors:

$$\begin{matrix} P_3 \\ P_3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{-18} \tag{1.2}$$

The number appended to the matrix is the Euler number of the manifold which for this case is -18. The manifold is of interest because it admits a freely acting Z_3 symmetry which may be realized by cyclically permuting the first three homogeneous coordinates of each space

$$(x^1, x^2, x^3, x^4) \times (y^1, y^2, y^3, y^4) \longrightarrow (x^2, x^3, x^1, \omega x^4) \times (y^2, y^3, y^1, \omega^2 y^4) \tag{1.3}$$

with $\omega^3 = 1$, $\omega \neq 1$. The group action on the manifold is fixed point free so the quotient manifold, which has points identified under the action of the group, is also a smooth Calabi-Yau manifold which has Euler number -6, corresponding to three generations of particles. The list of all CICY manifolds is a list of 7868 matrices which generalize (1.2) which are of the form

$$\begin{array}{c} P_{n_1} \\ P_{n_2} \\ \vdots \\ P_{n_F} \end{array} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{F1} & a_{F2} & \dots & a_{FN} \end{bmatrix} \tag{1.4}$$

the columns of which $a_{\alpha j}$, $j = 1, ..., F$ give the degrees of the polynomial p^α , $\alpha = 1, ..., N$ in the variables of each space. A prime motivation for the compilation of the list was to seek additional examples of Calabi-Yau manifolds corresponding to three generations of particles. All the matrices in the list have negative Euler number so these would be manifolds corresponding to Euler number -6. No spaces with Euler number -6 appear in the list. However there do appear a large number with Euler numbers that are divisible by 6, suggesting that, as with the Tian-Yau manifold it may be possible to find in the list a manifold $\mathcal{M}$ with Euler number -6k which admits a freely acting discrete isometry group $\mathcal{G}$ of order k. For such a manifold the quotient

$$\mathcal{M}_1 = \mathcal{M}/\mathcal{G} \tag{1.5}$$

would be a Calabi-Yau (CY) manifold with Euler number -6. The manifolds that have been constructed are given as embedded submanifolds. In such a situation there are two ways of defining a group action: one way would be to define the group action in an intrinsic way on the submanifold without referring to the ambient space. Ideally this is what one would do, however this is impractical for lack of technique. The second approach is to define an automorphism of the ambient space and consider the induced action on the submanifold. This is the approach we will

pursue in this article. However it must be borne in mind that group actions may exist that cannot be realized linearly on the homogeneous coordinates of the ambient space. Furthermore we do not discuss in this paper the technique of resolving singularities which occur when group actions are considered that have fixed point sets. This technique has already proved useful as two of the three constructions of three generation manifolds [2,3] employ this blow up procedure.

To the best of our knowledge conditions that are both necessary and sufficient for the existence even of such a free projective action are not known short of a concrete realization. It is however possible to formulate a number of *necessary* conditions for the existence of a free projective action. These conditions turn out to be very restrictive and the number of CY matrices that satisfy these conditions is small.

The necessary conditions that dramatically reduce the number of possibilities represent natural extensions of the Euler number divisibility condition. Recall that the Euler number of a manifold $\mathcal{M}^n$ can be described as the alternating sum of the dimensions of the deRham cohomology groups [1]

$$\chi(\mathcal{M}) = \sum_{p=0}^{n} (-1)^p b_p \tag{1.6}$$

and equivalently as an integral of the n'th Chern class

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} c_n. \tag{1.7}$$

The equivalence of these two expressions explains why $\chi(\mathcal{M})$ must be divisible by the order of a freely acting group. We see from (1.6) that both $\chi(\mathcal{M})$ and $\chi(\mathcal{M}/\mathcal{G})$ are integers and from (1.7) that the ratio of these integers is the order of the group. The tests that we apply to the matrices of the list to determine whether they admit

[1] Notation: By an upper index we denote the real dimension of a manifold, by a lower index we mean complex dimension, $\mathcal{M}^{2n} = \mathcal{M}_n$

freely acting groups concern the indices of certain differential operators. These indices are not topological quantities *on* $\mathcal{M}$ since they depend, in general, on the way that $\mathcal{M}$ is embedded in the ambient space. They do however share with χ the property that (modulo a number of caveats to be discussed in Section III) they are divisible by the order of a freely acting group. This follows from the fact that they admit representations analogous to (1.6) and (1.7). On the one hand they may be defined as the differences of the number of zero modes of certain differential operators and on the other hand they may be computed by means of certain integrals involving characteristic classes. They have the virtue of being directly computable from the degree matrix.

II. THE TESTS

The deRham cohomology groups and hence the Betti numbers are defined through the action of the exterior derivative d acting on p-forms. Let $\Lambda(\mathcal{M}^n)$ denote the Grassmann algebra of all p-forms, $0 \leq p \leq n$ of the manifold and by $H^*_{deRham}(\mathcal{M}^n)$ the sum of all cohomology groups $H^p_{deRham}(\mathcal{M}^n)$. The triple (Λ, H^*, d) is called an elliptic complex and the Euler number is its index.

The natural question is whether there are other elliptic complexes naturally associated with CICY manifolds. In fact there are. On every complex manifold the exterior derivative can be decomposed into a holomorphic and an antiholomorphic part; $d = \partial + \bar{\partial}$. There is an elliptic complex associated with the action of $\bar{\partial}$ on (0,q)-forms, $0 \leq q \leq n$, its index is the holomorphic Euler number

$$\begin{aligned} \chi^h(\mathcal{M}) &= \sum_{q=0}^{n}(-1)^q h^{(0,q)} \\ &= \int_{\mathcal{M}_n} td_n(\mathcal{T}), \end{aligned} \tag{2.1}$$

where $\mathcal{T}$ is the holomorphic part of the tangent bundle of $\mathcal{M}$ and td_n is the n'th Todd class in the total Todd class[4]

$$td(\mathcal{T}) = 1 + \frac{1}{2}c_1(\mathcal{M}) + \frac{1}{12}[c_2(\mathcal{M}) + c_1(\mathcal{M})^2] + \frac{1}{24}c_1(\mathcal{M})c_2(\mathcal{M}) + \dots \tag{2.2}$$

Furthermore for 4n-dimensional manifolds $\mathcal{M}^{4n}$ there is also the signature complex with index the Hirzebruch signature σ. Define a slightly modified Hodge operator that acts on p-forms

$$\begin{aligned} \#^p &= i^{p(p-1)+\frac{n}{2}} * \\ \#^p &: \Lambda \longrightarrow \Lambda \\ (\#^p)^2 &= 1 \end{aligned} \tag{2.3}$$

and let $\Lambda^{\pm}$ be the ± 1 eigenspaces of $\#^p$. Then the exterior derivative defines an operator

$$d + d^{\dagger} : C^{\infty}(\Lambda^{+}) \longrightarrow C^{\infty}(\Lambda^{-}) \tag{2.4}$$

whose index σ can be calculated by evaluating the integral

$$\sigma(\mathcal{M}) = \int_{\mathcal{M}} L(\mathcal{M}) \tag{2.5}$$

where $L(\mathcal{M})$ is the Hirzebruch L-polynomial

$$L(\mathcal{M}) = 1 + \frac{1}{3}p_1(\mathcal{M}) + \frac{1}{45}(7p_2(\mathcal{M}) - p_1(\mathcal{M})^2) + \dots \tag{2.6}$$

and the $p_i(\mathcal{M})$ are the Pontrjagin classes. Finally there is the spin complex with its $\hat{\mathrm{A}}$ genus. These are the four classical elliptic complexes.

At first sight it might appear that apart from the Euler number these indices are of no use to us since χ^h vanishes for every Calabi-Yau manifold, σ is defined only for $4n$-dimensions and on a Calabi-Yau manifold spinors may be represented by forms. However since by construction CICY manifolds are presented as embedded manifolds there is always a natural bundle associated with each CICY matrix, namely the normal bundle $\mathcal{N}$ of the manifold in the ambient space. By considering generalized quantities which take values in $\mathcal{N}$, for example, we are able to formulate nontrivial conditions with the indices.

Before discussing the tests themselves we wish to draw the attention to an important point. Not every CY manifold can be represented as a CICY manifold (no CY manifold that has positive Euler number can be represented in this form, for example) and if it can be represented as a CICY manifold it is reasonable to expect that in general this will only be the case for special values of the parameters of the manifold, *i.e.* for values of the parameters that lie in some subspace of

the parameter space. Also it is clear that the manifold will admit a freely acting symmetry, if at all, only for special values of its parameters. Thus situations can arise as in Figs.1 and 2.

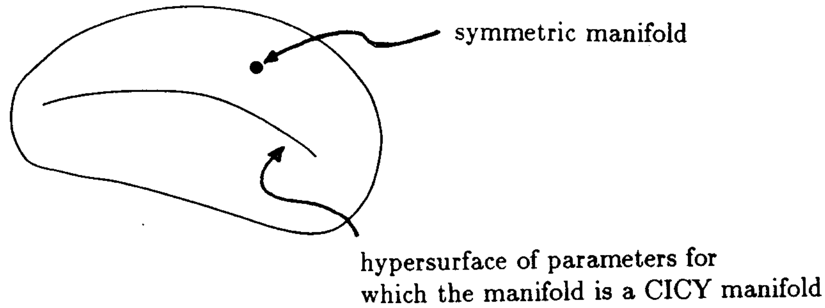


*Fig.*1

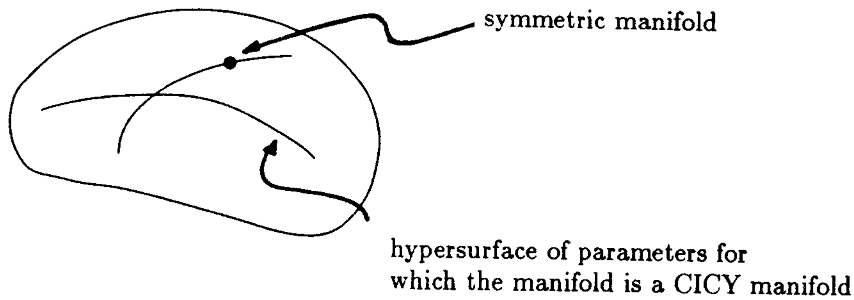


*Fig.*2

In Fig.1 there is a symmetric manifold diffeomorphic to a CICY manifold but the symmetric manifold cannot be realized as a CICY manifold. In Fig.2 a diffeomorphism class of manifolds admits a representation in terms of CICY matrices in two different ways and the symmetric manifold can be represented in one way but not in the other. It is in fact rather exceptional for the set of parameters for which the manifold may be realized as a CICY manifold to fill the entire parameter space, though this does occur for some simple cases such as Yau's manifold (2), i.e. by suitable choice of polynomials of the degrees indicated every member of the deformation class may be realized.

The reader is also cautioned that in the compilation of the list of matrices we have, as explained in [1], freely used a number of identities that relate CICY matrices representing *diffeomorphic* manifolds. In principle this can result in a change of parameters as in passing from one hypersurface to the other in Fig.2. A two dimensional example for which this is the case is provided by the K3 surface. It is well known that the K3 manifold admits a twenty parameter family of complex structures. It is easy to write down many different representations such as the following, which may all be related [5] via the identities presented in [1]

$$P_3\,[4]\,, \qquad P_4\,[2,3]\,, \qquad P_5\,[2,2,2]\,, \qquad \begin{array}{c} P_1 \\ P_2 \end{array}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad \begin{array}{c} P_1 \\ P_1 \\ P_1 \end{array}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \tag{2.7}$$

None of these representations, however, fills the parameter space. It is also well known that there are K3 manifolds that admit a freely acting Z_2 symmetry. It is easy to find such an action for $P_5[2,2,2]$, but it may be shown that no projectively induced Z_2 symmetry exists for the matrix

$$\begin{array}{c} P_1 \\ P_2 \end{array}\begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{2.8}$$

III. INDICES TWISTED WITH $\mathcal{N}$ AND $\mathcal{T}$

Returning to the four classical elliptic complexes we may consider the action of the differential operators on forms that take values in some vector bundle $\mathcal{E}$, *i.e.* we consider the operator acting on the bundle $\Lambda \otimes \mathcal{E}$. For the modified holomorphic Euler number we now find the expression

$$\chi^h(\mathcal{E}) = \int_{\mathcal{M}} td(\mathcal{T}) \wedge ch(\mathcal{E}) \tag{3.1}$$

while for the modified signature we find

$$\sigma(\mathcal{E}) = \int_{\mathcal{M}} L(\mathcal{M}) \wedge \widetilde{ch}(\mathcal{E}) \tag{3.2}$$

where $\widetilde{ch}$ denotes the Chern character with the curvature form $\mathcal{R}$ replaced by $2\mathcal{R}$. Twisting the deRham complex and the spin complex gives nothing new.

Now applying these results to CICY manifolds the natural bundle to twist with is the normal bundle $\mathcal{N}$ of the embedded manifold. It is also possible to twist with arbitrary tensor powers $\mathcal{N}^{\otimes k}$ and with tensor powers $\mathcal{T}^{\otimes l}$ of the tangent bundle. Thus it is possible to write down an infinite number of indices. However, these are not all independent and we shall see that for $k > 3$ and $l > 1$ no new conditions emerge. The twisted holomorphic Euler number gives rise to the most stringent tests so we begin our discussion with this index.

The total Todd class of a Calabi-Yau manifold is given by

$$td(\mathcal{M}) = 1 + \frac{1}{12} c_2(\mathcal{M}). \tag{3.3}$$

With the aid of the relations of Section II of [1] we find (recall that N denotes the number of polynomials)

$$
\begin{aligned}
ch(\mathcal{N}^k \otimes \mathcal{T}^l) = \; & 3^l N^{k-3}\Big(N^3 + kN^2 c_1(\mathcal{N}) \\
& + \frac{1}{2}k(k-1)Nc_1^2(\mathcal{N}) + \frac{1}{6}k(k-1)(k-2)c_1^3(\mathcal{N})\Big) \\
& + 3^l k N^{k-2} ch_2(\mathcal{N})\Big(N + (k+1)c_1(\mathcal{N})\Big) \\
& - 3^{l-1} l N^{k-1} c_2(\mathcal{M})\Big(N + kc_1(\mathcal{N})\Big) \\
& + 3^l k N^{k-1} ch_3(\mathcal{N}) + \frac{1}{2}3^{l-1} l N^k c_3(\mathcal{M}).
\end{aligned} \tag{3.4}
$$

Thus, in virtue of (2.1),we have

$$
\begin{aligned}
\chi^h(\mathcal{N}^k \otimes \mathcal{T}^l) = \; & 3^l k N^{k-1} \chi^h(\mathcal{N}) \\
& + 3^{l-1} N^{k-3} \int \Big[\frac{1}{2}k(k-1)(k-2)c_1^3(\mathcal{N}) - klN^2 c_1(\mathcal{N})c_2(\mathcal{M}) \\
& \qquad + \frac{1}{2} l N^3 c_3(\mathcal{M}) + 3k(k-1)Nc_1(\mathcal{N})ch_2(\mathcal{N})\Big]
\end{aligned} \tag{3.5}
$$

where

$$
\chi^h(\mathcal{N}) = \int \Big[\frac{1}{12}c_2(\mathcal{M})c_1(\mathcal{N}) + ch_3(\mathcal{N})\Big] \tag{3.6}
$$

From these expressions it follows that the only independent indices are $\chi^h(\mathcal{N}^k \otimes \mathcal{T}^l)$ for $(k,l) \in \{(0,1);(1,0),(2,0),(3,0)\}$. In other words, if all the indices corresponding to these values of (k,l) are divisible by a given integer then the index is divisible by that integer for all (k,l).

The analogous expressions for the signature complex are

$$
\begin{aligned}
\sigma(\mathcal{N}^k \otimes \mathcal{T}^l) = \; & 3^l k N^{k-1} \sigma(\mathcal{N}) \\
& + 3^{l-1} 8 N^{k-3} \int \Big[\frac{1}{2}k(k-1)(k-2)c_1^3(\mathcal{N}) + \frac{1}{2} l N^3 c_3(\mathcal{M}) \\
& \qquad - klN^2 c_1(\mathcal{N})c_2(\mathcal{M}) + 3k(k-1)Nc_1(\mathcal{N})ch_2(\mathcal{N})\Big]
\end{aligned} \tag{3.7}
$$

with

$$\sigma(\mathcal{N}) = \int \Big[-\frac{4}{3} c_1(\mathcal{N}) c_2(\mathcal{M}) + 8 ch_3(\mathcal{N}) \Big]. \tag{3.8}$$

Comparing this result with that for the twisted Euler number leads to the conclusion that the only new condition is the divisibility of $\sigma(\mathcal{N} \otimes \mathcal{T})$.

On subjecting the list of 7868 Calabi-Yau matrices to these tests we find that only 189 survive.

IV. FURTHER TESTS

A list of 189 matrices is certainly a lot shorter than a list of 7868 but it is still impractical to analyse the remaining matrices by hand especially in view of the fact that inspection reveals that the great bulk of the matrices do not appear to admit freely acting groups of the required order. Thus we seek additional tests to reduce the list still further. These tests involve twisting with the individual normal bundles $\mathcal{N}_\alpha$, $\alpha = 1, \ldots, N$ corresponding to each polynomial p^α and with the hyperplane bundles $\mathcal{O}_j$, $j = 1, \ldots, F$ of each factor space [2]. It is also possible to twist with arbitrary tensor products of these. There are however a number of qualifications that must be discussed in this regard.

Firstly there is the possibility that the elements of a symmetry might interchange the coordinates of some of the projective spaces, we shall refer to such a group action as being *external* to the factor spaces in question [3]. There are many examples of this in the list of 189 matrices. We present two here. The first has a mixed symmetry that is a symmetry that acts externally on some factors and internally on others, while the second has a symmetry that is purely external. The structure of these matrices is most easily understood in terms of the diagrams introduced by Green and Hübsch[7]. An open circle represents a projective space, a dot represents a polynomial, the number of lines joining a dot to a circle denotes the degree of the polynomial in the homogeneous variables of the relevant space. The dimension of the projective space represented by a given circle is one less than

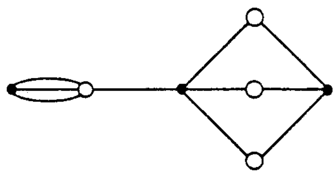
[2] We are grateful to B.Greene and to M.Eastwood for independently suggesting these tests to us.

[3] In a recent paper[6] Aspinwall, Greene, Kirklin and Miron (AGKM) report a search for three generation spaces that is parallel to ours in several respects. These authors assert that if a symmetry acts externally on some of the factor spaces then the symmetry inevitably has fixed points. This assertion, which is based on a dimension counting argument, is false as can be seen from the counterexamples presented in the following. Technically the proof of AGKM fails because they consider only CICY matrices of a specific form rather than of a more general type. Moreover these authors neglect the possibility that the symmetry may act externally on some of the factors and internally on the others.

the total number of lines emanating from it.

(i) Consider the space

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_3 \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{-36}$$



(4.1)

There is a fixed point free Z_3 which cyclically permutes the three P_1 factors. Let the coordinates be x_a^i, i=1,2 , a=1,2,3, let t^m, m=1,...,4 be the coordinates of the P_3 and let the generator g of the Z_3 have the following action

$$\begin{aligned} g :& x_a^i \longrightarrow x_{a+1}^i \\ g :& (t^a, t^4) \longrightarrow (t^{a+1}, t^4) \end{aligned} \tag{4.2}$$

where a is understood as being identified modulo 3. Thus g cyclically permutes the P_1's and also the first three coordinates of the P_3. If the polynomials are chosen of the form

$$\begin{aligned} p^1 &= \sum_a A_{ijk} x_a^i x_{a+1}^j x_{a+2}^k t^a + B_{ijk} x_1^i x_2^j x_3^k t^a \\ p^2 &= C_{ijk} x_1^i x_2^j x_3^k \\ p^3 &= \sum_a (t^a)^3 - 3(t^4)^3 \end{aligned} \tag{4.3}$$

with B_{ijk} and C_{ijk} symmetric tensors, then g is a symmetry of the manifold. Fixed points of g satisfy

$$\begin{aligned} x_a^i &= x^i, \ a = 1,2,3 \\ t^m &= \begin{cases} (1, \omega, \omega^2, 0), & \text{if } \omega^3 = 1, \omega \neq 1; \\ (t^1, t^1, t^1, t^4). \end{cases} \end{aligned} \tag{4.4}$$

The point $(1, \omega, \omega^2, 0)$ does not satisfy $p^3 = 0$. By requiring the other value of t^m to satisfy $p^3 = 0$ we find

$$t^m = (1, 1, 1, \omega), \quad \omega^3 = 1 \tag{4.5}$$

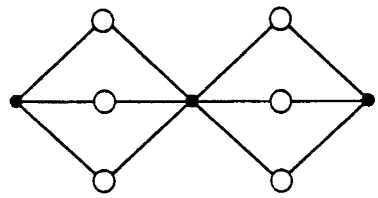
and the remaining constraints become

$$\begin{aligned} 3A_{ijk}x^ix^jx^k + \omega B_{ijk}x^ix^jx^k &= 0 \\ C_{ijk}x^ix^jx^k &= 0 \end{aligned} \tag{4.6}$$

these are two cubic equations acting in a P_1. A simple dimension count therefore does not rule out this symmetry.

(ii) The manifold

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{-72} \tag{4.7}$$



admits an external freely acting Z_3 symmetry. Let us denote the coordinates of the three P_1's on the left of the diagram by x_a^i and the coordinates of the P_1's on the right by y_a^i. If we choose the polynomials to be

$$\begin{aligned} p^1 &= A_{ijk}x_1^ix_2^jx_3^k \\ p^2 &= B_{lmn}y_1^ly_2^my_3^n \\ p^3 &= C_{ijklmn}x_1^ix_2^jx_3^ky_1^ly_2^my_3^n \end{aligned} \tag{4.8}$$

where the coefficients have the symmetries

$$A_{ijk} = A_{(ijk)} \quad , \quad B_{lmn} = B_{(lmn)} \tag{4.9}$$

and

$$C_{ijklmn} = C_{jkimnl} \tag{4.10}$$

then

$$\begin{aligned} g &: x_a^i \longrightarrow x_{a+1}^i \\ g &: y_a^l \longrightarrow y_{a+1}^l \end{aligned} \tag{4.11}$$

is a symmetry of the manifold. The fixed point set of g in the ambient space is $P_1 \times P_1$

$$x_a^i = x^i \quad , \quad y_a^l = y^l \quad , \quad a = 1, 2, 3. \tag{4.12}$$

The constraints become three equations acting in $P_1 \times P_1$ and we have a similar situation as in (i).

The existence of these external symmetries complicates the application of the divisibility tests. Consider for example the manifold of Fig.3.

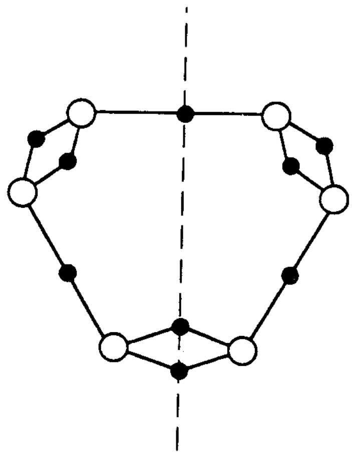


Fig.3

which has Euler number -12. *A priori* there might exist an external Z_2 symmetry corresponding to a reflection of the diagram in the dotted line. Such a reflection interchanges some of the polynomials. If p^α and p^β, say, are interchanged by this operation then one may only test indices in which $\mathcal{N}_\alpha$ and $\mathcal{N}_\beta$ appear symmetrically. In this particular example it is easy to see that this Z_2 has fixed points but in other examples this might not be so. In priciple one could draw all the diagrams and by inspection see which external symmetries might occur. A conservative test however is to twist only with direct sums of the $\mathcal{N}_\alpha$'s corresponding to columns of the matrix

that contain the same components up to permutation. Although suboptimal this test is quite effective in reducing the number of possibilities.

We have been led to consider the possibility that a symmetry operation may permute some of the polynomials as the result of a consideration of external symmetries. Note however that it is possible for a symmetry to be purely internal and yet it may still permute the p^α.This is illustrated by the manifold $P_7[2,2,2,2]$ with polynomials

$$p^\alpha = \frac{1}{2}\sum_{k=1}^{8} z_k^2 i^{k\alpha} + \sum_{k=1}^{8} z_{k-1}z_{k+1}(\lambda i^{k\alpha} + \mu i^{-k\alpha}) + \frac{\delta}{2}\sum_{k=1}^{8} z_{k-2}z_{k+2}i^{k\alpha},\ i^2 = -1. \tag{4.13}$$

It has been shown [8] that this manifold admits a freely acting $Z_4 \times Z_8$ symmetry generated by

$$g : z_k \longrightarrow i^k z_k \tag{4.14}$$

$$h : z_k \longrightarrow z_{k+1} \tag{4.15}$$

The action of g permutes the polynomials

$$g : p^\alpha \longrightarrow p^{\alpha+2} \tag{4.16}$$

where α is to be understood modulo four. This prompts consideration of a more general situation. Suppose, as in the present case, that the symmetry operations $g \in \mathcal{G}$ permute the constraints

$$p^\alpha \longrightarrow L^\alpha{}_\beta(g)p^\beta. \tag{4.17}$$

Suppose also that our indices are twisted with just one $\mathcal{N}_\gamma$ then the full group $\mathcal{G}$ may not be a symmetry of $\Lambda \otimes \mathcal{N}_\gamma$. Nevertheless since the permutation (4.17) can only be between polynomials that have the same degree it follows that the conservative test outlined above of twisting only with the direct sums of the $\mathcal{N}_\alpha$ corresponding to columns of the matrix that contain the same elements up to permutation covers this case also. There is a further test that may be applied to indices twisted with the

individual $\mathcal{N}_\gamma$ since the polynomials may always be chosen such that the $L^\alpha{}_\beta(g)$ for $g's$ in any given cyclic subgroup act diagonally on the p^α. It is therefore appropriate to test the indices only for divisibility by the orders of the cyclic subgroups of $\mathcal{G}$ [4]. Put another way: the highest common factors of the indices, for the cases where some of the p^α might be permuted by the elements of $\mathcal{G}$, yields information about the cyclic subgroups of $\mathcal{G}$. Although this is in principle an independent test we do not employ it since it turns out not to be very effective in reducing the list. We turn now to a discussion of indices twisted with the individual hyperplane

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4 AGKM assert more: namely that the p^α may be chosen in such a way that abelian subgroups of $\mathcal{G}$ are symmetries of $\Lambda \otimes \mathcal{N}_\gamma$. This is in general false. The argument of AGKM is based on the assumption that the $L^\alpha{}_\beta(g)$ form a representation of $\mathcal{G}$ and hence the $L^\alpha{}_\beta$'s corresponding to $g's$ in an abelian subgroup of $\mathcal{G}$ would be simultaneously diagonalizable. The $L^\alpha{}_\beta$, however, do not in general form a representation of $\mathcal{G}$, they may form a representation of another group $\mathcal{G}'$ which may be nonabelian even if $\mathcal{G}$ is abelian. To see how this can arise consider the two polynomials

$$p^\alpha = x^\alpha(x^1y^1 + x^2y^2) + ((x^1)^2 + (x^2)^2)y^\alpha, \quad \alpha = 1,2,$$

which may be thought of as being of degree (2,1) in the coordinates of $P_1 \times P_1$. Let s and t act on the coordinates in the following way

$$\begin{aligned} s :&(x^1, x^2) \times (y^1, y^2) \longrightarrow (x^1, -x^2) \times (y^1, -y^2) \\ t :&(x^1, x^2) \times (y^1, y^2) \longrightarrow (x^2, x^1) \times (y^2, y^1) \end{aligned}$$

s and t generate a group $Z_2 \times Z_2$. However their respective action on the polynomials

$$p^\alpha \longrightarrow S^\alpha{}_\beta p^\beta, \quad p^\alpha \longrightarrow T^\alpha{}_\beta p^\beta$$

is represented by the matrices

$$S^\alpha{}_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

S and T do not commute. In fact since they are two of the Pauli matrices they generate the *nonabelian* group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternions. For the case of $P_7[2\ 2\ 2\ 2]$ presented above it is easy to check that, acting on the p^α, the $Z_4 \times Z_8$ symmetry leads to a nonabelian sixteen element group.

bundles $\mathcal{O}_j$ and powers thereof. If the twisting is performed with the bundle [5] $\mathcal{O}_1(d_1) \otimes \mathcal{O}_2(d_2) \otimes \ldots \otimes \mathcal{O}_F(d_F)$ this amounts to considering zero-modes that have coefficients that transform like polynomials of degree $(d_1, d_2, \ldots, d_F)$. Consider for example the manifold $P_4[5]$ with the simplest polynomial

$$p = (z_1)^5 + (z_2)^5 + (z_3)^5 + (z_4)^5 + (z_5)^5 \tag{4.18}$$

then this manifold admits a freely acting $Z_5 \times Z_5$ symmetry [9] generated by

$$\begin{aligned} A: z_m &\longrightarrow \alpha^m z_m, \quad \alpha = e^{\frac{2\pi i}{5}} \\ C: z_m &\longrightarrow z_{m+1} \end{aligned} \tag{4.19}$$

If A and C are regarded as matrices acting on the coordinates then they do not commute

$$ACA^{-1}C^{-1} = \alpha \tag{4.20}$$

As automorphisms of P_4 these maps do, of course, commute since the multiplication of the coordinates by a common factor is of no consequence. However, if we compute an index twisted with $\mathcal{O}(1)$ say, then we are dealing with quantities that transform like linear polynomials and there is no linear polynomial invariant under both A and C. This is a consequence of the commutation relation (4.20) from which it follows that the matrix representation of A and C cannot be simultaneously diagonalized. There is a linear polynomial invariant under A and another invariant under C, but none invariant under both. Thus the invariance of $\Lambda \otimes \mathcal{O}(1)$ is Z_5 and not the full group $Z_5 \times Z_5$.

More generally, when we apply divisibility tests to the indices twisted by the hyperplane bundles we shall require the indices to be divisible by the orders of the cyclic subgroups of $\mathcal{G}$. By choice of scale we may always ensure that the matrices

[5] Notation: $\mathcal{O}_j(d)$ denotes the d'th tensor power of the hyperplane bundle of P_{n_j}.

representing the action of a cyclic subgroup of $\mathcal{G}$ of order m commute, and that the m'th power of the matrix that represents the generator is the identity. Provision must also be made for the possibility that $\mathcal{G}$ involves external symmetries. A conservative test that does not require inspection of the diagram is to consider only indices of operators that are twisted by the same powers of the hyperplane bundle for each of the projective spaces of a given dimension. Thus for the manifold (4.1) for example we test with $\mathcal{O}(k) \otimes \mathcal{O}(k) \otimes \mathcal{O}(k) \otimes \mathcal{O}(l)$.

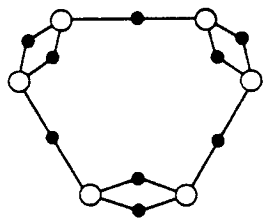
The above tests reduce the list of candidate matrices from 189 to 21. To the remaining 21 more stringent tests can be applied after drawing their diagrams to see which external symmetries are possible.

V. FROM 21 MATRICES TO 4

Presented below is the list of the 21 matrices that survive the tests that we have described. Together with their respective diagrams. The diagrams have been drawn such as to display the greatest possible symmetry [6].Following the list we describe the tests to which the matrices are subject.

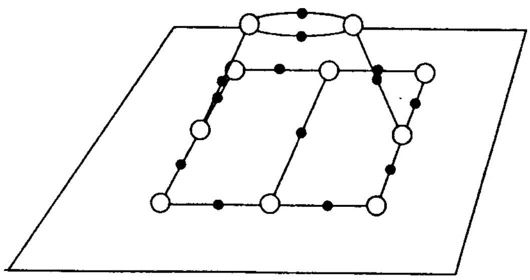
(5.1)

$$\begin{array}{c} P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{-12}$$



(5.2)

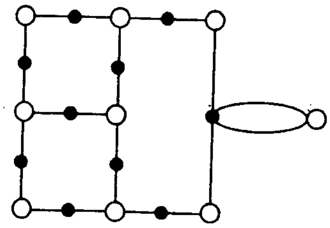
$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{-12}$$



[6] This has been done by inspection on a case by case basis. This is not a wholly satisfactory procedure since it is difficult to know when some greater symmetry might exist.The reader who draws a few diagrams directly from the matrices will appreciate that it is easy to draw the diagrams in such a way that very little symmetry is manifest.

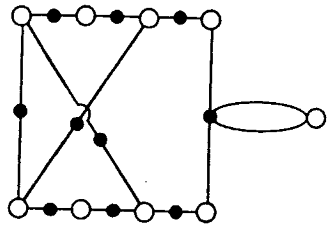
(5.3)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \left[\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]_{-12}$$



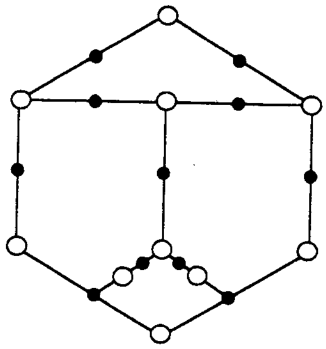
(5.4)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \left[\begin{array}{cccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]_{-12}$$



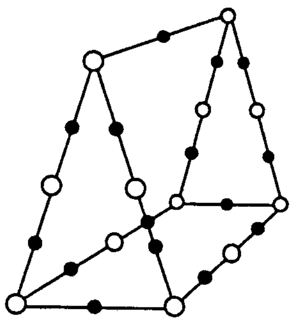
(5.5)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \left[\begin{array}{ccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]_{-12}$$



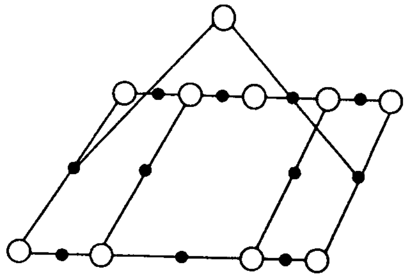
(5.6)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \left[\begin{array}{ccccccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{-12}$$



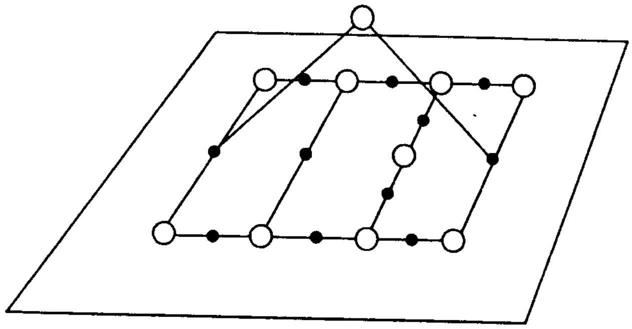
(5.7)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0\\ 1&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&1&1&0\\ 0&0&0&0&1&0&0&1&1&0&0\\ 0&0&0&1&0&0&0&0&0&1&1\\ 0&1&1&0&0&0&0&0&0&0&1 \end{bmatrix}_{-12}$$



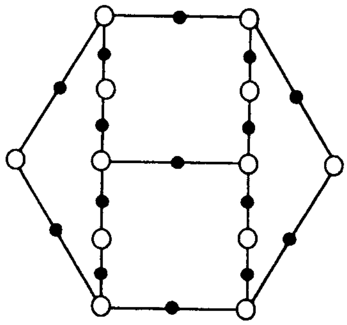
(5.8)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0\\ 1&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&1&1&0&0\\ 0&0&1&0&1&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&1&0&1\\ 0&1&0&0&0&0&0&0&0&1&1 \end{bmatrix}_{-12}$$



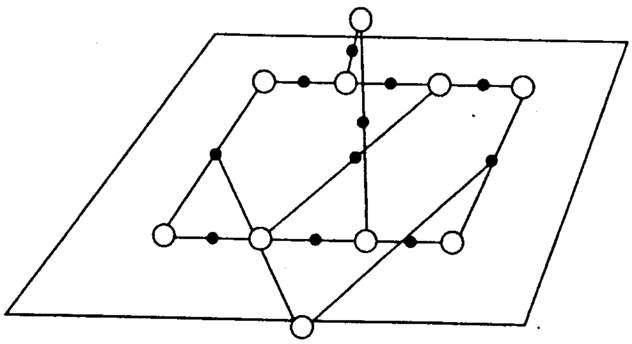
(5.9)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_2\\ P_2\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&1&0&1&0&0\\ 1&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&1&0&0&0&0&0&0&0&0&1 \end{bmatrix}_{-12}$$



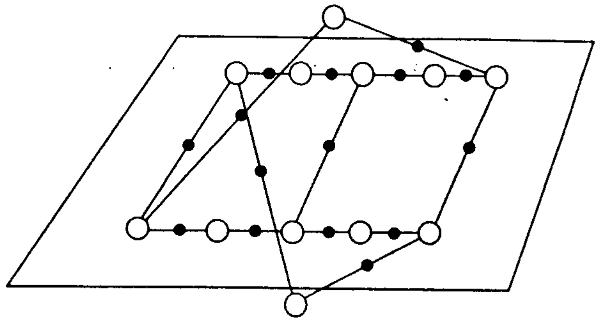
(5.10)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&1&0&0&0\\ 1&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0\\ 1&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&1&1\\ 0&0&0&1&0&1&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&1&0&1 \end{bmatrix}_{-12}$$



(5.11)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{-12}$$

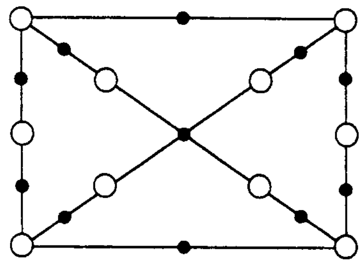


(5.12)

$$\begin{array}{c} P_3 \\ P_3 \end{array} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{-18}$$

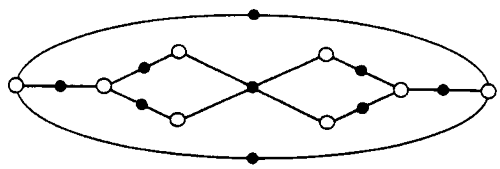
(5.13)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{-24}$$



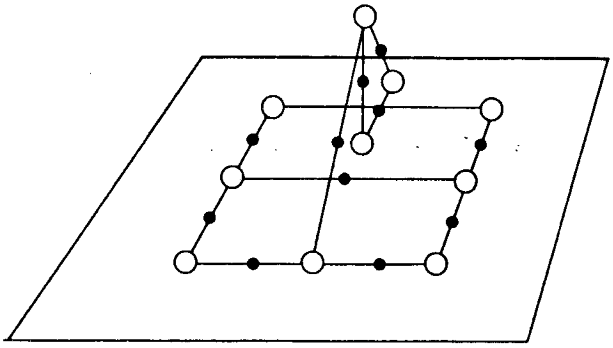
(5.14)

$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_2 \\ P_2 \\ P_2 \\ P_2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{-24}$$



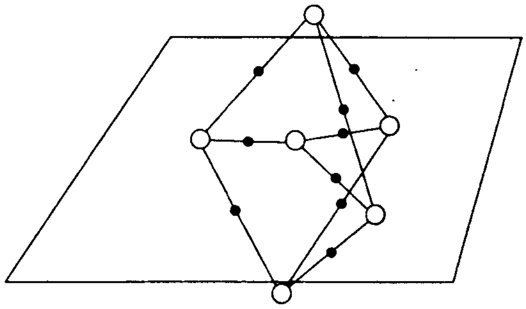
(5.15)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&1&0&0&0\\ 1&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&1&1&0&0&0&1&0 \end{bmatrix}_{-24}$$



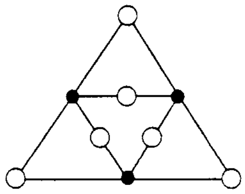
(5.16)

$$\begin{array}{c} P_2\\ P_2\\ P_2\\ P_2\\ P_2\\ P_2 \end{array}\begin{bmatrix} 1&1&1&0&0&0&0&0&0\\ 1&0&0&1&1&0&0&0&0\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&0&0&0&0&1&1\\ 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&1&0&1&0&1 \end{bmatrix}_{-36}$$



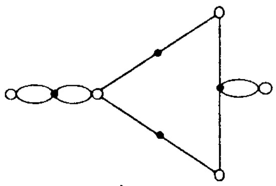
(5.17)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_1\\ P_1 \end{array}\begin{bmatrix} 0&1&1\\ 0&1&1\\ 1&1&0\\ 1&1&0\\ 1&0&1\\ 1&0&1 \end{bmatrix}_{-48}$$



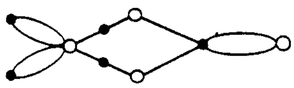
(5.18)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_3 \end{array}\begin{bmatrix} 1&1&0&0\\ 1&0&1&0\\ 0&0&0&2\\ 2&0&0&0\\ 0&1&1&2 \end{bmatrix}_{-48}$$



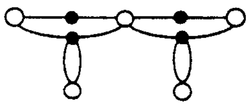
(5.19)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_5 \end{array}\begin{bmatrix} 1&1&0&0&0\\ 1&0&1&0&0\\ 2&0&0&0&0\\ 0&1&1&2&2 \end{bmatrix}_{-48}$$



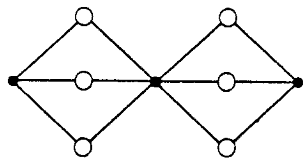
(5.20)

$$\begin{array}{c} P_1\\ P_1\\ P_1\\ P_1\\ P_3 \end{array}\begin{bmatrix} 1&1&0&0\\ 0&0&1&1\\ 2&0&0&0\\ 0&0&2&0\\ 1&1&1&1 \end{bmatrix}_{-48}$$



(5.21)

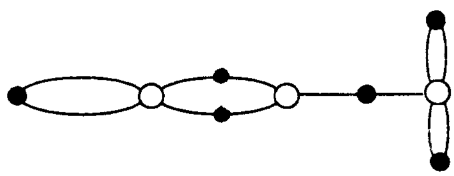
$$\begin{array}{c} P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \\ P_1 \end{array}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{-72}$$



Consider first the manifold (5.1), which we have encountered previously, *a priori* there might exist an external symmetry as in Fig.3 corresponding to a reflection in a vertical axis. Let the homogeneous coordinates of the three spaces on the left of the diagram be ${x_a}^i$ with i a coordinate index and a an index that labels the spaces, and let ${y_a}^i$ be the corresponding coordinates on the right of the diagram. Fixed points of the symmetry satisfy

$$ {x_a}^i = {y_a}^i, \quad i = 1,2,3; \; a = 1,2,3. \tag{5.22}$$

The symmetry also identifies the three polynomials on the right with the three on the left. The fixed point set corresponds therefore to the diagram



Six equations act in $P_2 \times P_2 \times P_2$ and these always have a solution. Thus no external freely acting Z_2 symmetry exists in this case. This being so it suffices to test for

purely internal symmetries with arrays

$$\mathcal{H}[a,b,c,d,e,f] \qquad \mathcal{N}[i,i,j,k,l,l,m,m,n] \tag{5.23}$$

These arrays describe the indices that are used to test each matrix. An $\mathcal{H}$ array

$$\mathcal{H}[a,b,\ldots,f]$$

means that twisted indices are computed and tested for the bundle

$$\mathcal{O}(a) \otimes \mathcal{O}(b) \otimes \ldots \otimes \mathcal{O}(f)$$

with independent values assigned to distinct arguments. While an $\mathcal{N}$ array

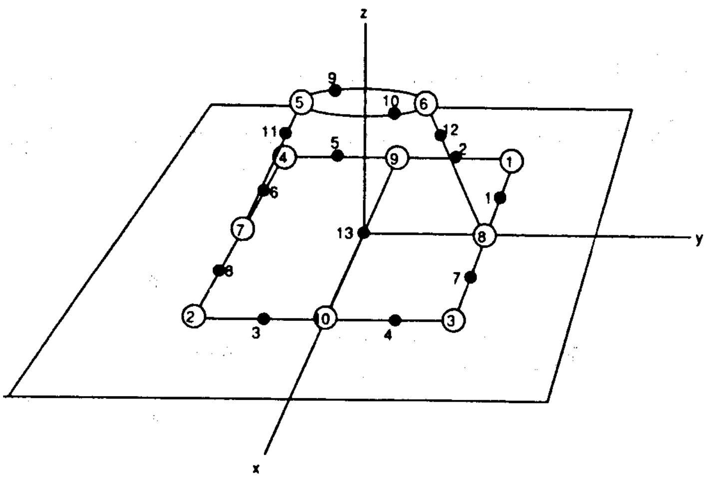
$$\mathcal{N}[i,j,i,j,k],$$

for example, means that twisted indices were computed for the bundles

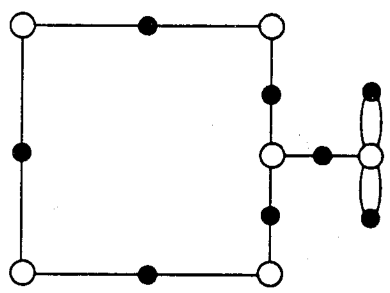
$$(\mathcal{N}_1 \oplus \mathcal{N}_3)^i \otimes (\mathcal{N}_2 \oplus \mathcal{N}_4)^j \otimes (\mathcal{N}_5)^k$$

with independent values assigned to the arguments i,j,k. On computing the indices corresponding to the arrays (5.23) it is found that some of them are odd. Thus we learn that for this manifold there are no freely acting internal symmetries either.

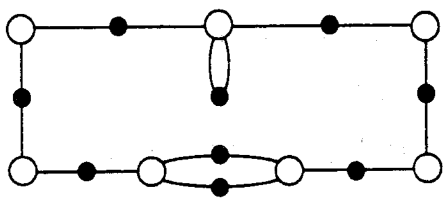
Matrix (5.2) is a more complicated example. The diagram has three reflection symmetries



corresponding to reflections in the x-z plane, the y-z plane and the z -axis. Let us deal with these three cases in turn. The situation with regard reflection in the x-z plane is similar to the previous case. The fixed point diagram is

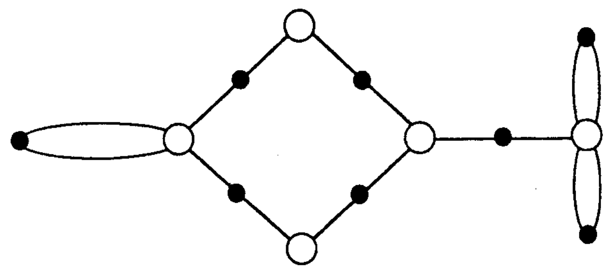


We have eight equations acting in an eight dimensional space so again such a symmetry has fixed points. Reflection in the y-z plane leaves four of the spaces fixed so that this Z_2 could be part of a mixed symmetry that acts internally in these four P_2's. A Z_2 acting within a P_2 has a fixed point set at least as large as a P_1 so the fixed point diagram for this case is



here nine equations act in a space of eight dimensions. So this Z_2 might be fixed point free.

Finally reflection in the z−axis has a fixed point diagram



eight equations act in a space of dimension eight so this symmetry always has fixed points. We have learnt that it is sufficient to allow for a Z_2 that reflects in the y-z plane in other words it is sufficient to test with

$$\mathcal{H}\,[a,b,a,b,c,d,e,f,g,g,]\,,\quad \mathcal{N}\,[i,j,k,j,k,l,i,l,m,n,p,q,r]\,. \tag{5.24}$$

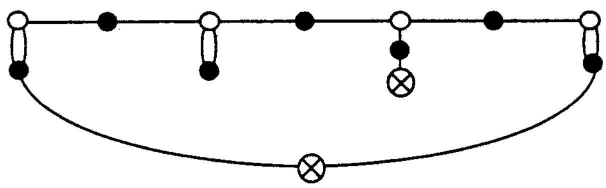
Some of the indices corresponding to these arrays are odd so we conclude that there is no freely acting Z_2 symmetry for this manifold.

Proceeding in this manner we find that the bulk of the matrices are eliminated. The only survivors being (5.8), (5.12), (5.17) and (5.18).

VI. THE LAST FOUR

Of the four survivors the second, (5.12), is the covering manifold of the Z_3-orbit manifold described in [2] which motivated the study of CICY manifolds. We consider the other three in turn.

(i) The matrix (5.8) describes a class of manifolds with Euler number $\chi = -12$. The diagram associated to this matrix suggests an obvious Z_2 symmetry corresponding to the reflection in the (y,z) plane. The symmetry may also act internally to spaces 2 and 4. The fixed point diagram for this symmetry is



Here seven equations act in a six dimensional space if we take account of the fact that a Z_2 symmetry acting internally on spaces 2 and 4 will have fixed point sets of dimension zero (denoted by $\otimes$). Since this symmetry may act freely the matrix was tested with the arrays

$$\mathcal{H}\,[a,b,a,c,d,d,e,e,f]$$

$$\mathcal{N}\,[i,j,k,j,l,l,m,k,n,n,p]$$

All these indices turn out to be even. If however we enquire about the possibility of a purely internal Z_2 and test with $\mathcal{H}$ and $\mathcal{N}$ arrays with independent arguments then some of these indices are odd. Thus no purely internal freely acting Z_2 symmetry exists for this manifold. So if a freely acting Z_2 exists it must be the mixed symmetry discussed above. A more detailed analysis however shows that this action in fact has fixed points as we will show now.

Without loss of generality we can assume that the internal Z_2 actions on the P_1's are given by

$$g : P_1 \longrightarrow P_1$$

$$(z^0, z^1) \mapsto (z^0, -z^1)$$

Then the fixed points of this action are $f_1 = (1,0)$ and $f_2 = (0,1)$. For the first polynomial

$$p^1 = A(x_3)z^0 + B(x_3)z^1$$

to be invariant we must drop the B-term. This however is fatal since p^1 vanishes identically on f_2 and we are left with six polynomials on a six dimensional space.

(ii) The second survivor in the list of 21 is (5.17). Here the situation is more involved as there are three other diffeomorphic representations related to each other by the identity

$$P_1 \times P_1 \hookrightarrow P_3[2]$$

this is the well known Segre embedding. These realizations are all equivalent and we concentrate here on the matrix (5.17) The situation is the following. There are several symmetries of the diagram which might form part of a group of order eight. However by an enumeration of possibilities we find that the only one that might act freely is a Z_2 symmetry which acts internally on the four spaces connected by a given polynomial and which interchanges the other two spaces. An example would be a symmetry acting internally to spaces 1,2,3,4 and interchanging spaces 5 and 6 in (6.2). Thus it suffices to compute indices corresponding to the arrays

$$\mathcal{H}\,[a, b, c, d, e, e]$$

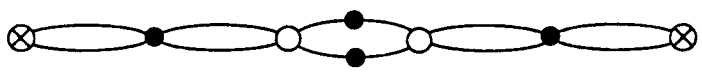
$$\mathcal{N}\,[i, j, k]\,.$$

The indices twisted with the hyperplane bundle have two as their highest common factor. We learn that if a freely acting symmetry group G exists then its largest cyclic subgroup is of order two. There are five discrete groups of order eight

$$Z_2 \times Z_2 \times Z_2 \ , \quad Z_2 \times Z_4 \ , \quad Z_8 \ , \quad Q_4 \ , \quad D_4$$

(D_4 is the symmetry group of the square). Apart from the first they all have Z_4 as a subgroup. Thus if G exists it must be $Z_2 \times Z_2 \times Z_2$. The indices corresponding to the $\mathcal{N}$-array are all divisible by eight so we learn nothing new from these.

(iii) The last example, (5.18) has also Euler number $\chi = -48$ but is topologically distinct from the manifold (5.17) described in (ii). The obvious candidate for a fixed point free external action is a reflection with respect to the vertical axis. Furthermore we allow for an internal action in the spaces 3,4, and 5. The fixed point diagram of this mixed symmetry is



where we take into account that an internal Z_2 action in P_3 has a minimal fixed point set isomorphic to P_1. Here we have four equations in a two dimensional space and therefore the symmetry may act freely. Because of this it was tested with the arrays

$$\mathcal{H}\,[a, a, b, c, d]$$

$$\mathcal{N}\,[i, j, j, k]$$

It turns out that the hyperplane bundle twists lead to numbers whose biggest common factor is two, so that the biggest cyclic subgroup possible is Z_2. Therefore if a fixed point free action exists it can, again, only be G=$Z_2 \times Z_2 \times Z_2$. The normal bundle tests all lead to numbers divisible by eight and therefore give no new restrictions.

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